THE OCTAHEDRAL GROUP

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In geometry, any polyhedron is associated with a dual figure, where the vertices of one correspond to the faces of the other and the edges between pairs of vertices of one correspond to the edges between pairs of faces of the other. Since duality clearly preserves symmetries, studying the symmetries of a figure is the same as studying the symmetries of its dual. We shall in this paper study the Isometry group of the Octahedron, by studying the one of its dual, the cube.
Let $X := \{-1, 1\}^3 \subset \mathbb{R}^3$ be the set of the eight points of coordinates $(\pm1, \pm1, \pm1)$. These points form a cube, and shall be noted $A, B, C, D, A', B', C', D'$ where A', B', C', D' are, respectively, inversions of A, B, C, D with respect to the origin O.

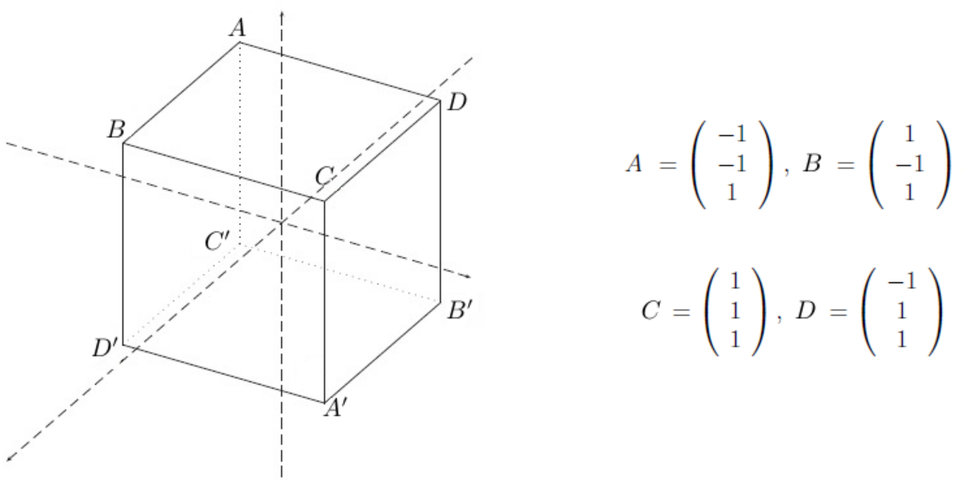

Figure 1

Let $G = \text{Iso}(X)$ be the Isometry group of X. As $g(O) = O$ for all $g \in G$, O is a fixed point of $\text{Iso}(X)$ (being the centroid of the cube). We get that $G \subset GL_3(R)$, and we shall then identify every element $g \in G$ by its matrix the standard basis of $\mathbb{R}^3$. (Moreover, G is a subgroup of $O_3(\mathbb{R})$, being the group of origin preserving isometries.)

Consider the set $\lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of axes of coordinate $\lambda_i = \mathbb{R}e_i$, $\forall i \in \{1, 2, 3\}$.
Let $T := \{\pm e_1, \pm e_2, \pm e_3\}$ be the set of points of $\mathbb{R}^3$ with just one non-zero coordinate, equal to ±1.
T is G-stable, being the set of centroids of each face of the cube. As λ is the set of straight lines $\mathbb{R}\,\overrightarrow{OM}$, with $M \in T$, λ is G-stable too. That yields to an action of G on the set λ, and thus a morphism

$$\psi : G \longrightarrow \mathfrak{S}(\lambda) \cong \mathfrak{S}_3\,.$$

Consider the subgroup H of G of permutation matrices

$$H := \{P_\sigma \in GL_3(\mathbb{R}), \sigma \in \mathfrak{S}_3\} \quad \text{with } P_\sigma(e_i) = e_{\sigma(i)}\,.$$

It is clear that the restriction of $\psi|_H : H \longrightarrow \mathfrak{S}_3$ is an isomorphism, and hence ψ is surjective. Now, consider the second subgroup

$$K := \{\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in GL_3(\mathbb{R}), \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}\} \cong (\{-1,1\}^3, \times) \cong (\mathbb{Z}_2^3, +)$$

By stability of T under the action of G, we have $K \subset \ker\,\psi$. In the meantime, for all $g \in \ker\psi$ and all $i \in \{1,2,3\}$, g fixes each straight line $\lambda_i = \mathbb{R}e_i$. Thus it is a diagonal matrix $g = \text{diag}(x_i)$, with $x_i \in \mathbb{R}^*$. Since $g \in O_3(\mathbb{R})$, $x_i = \pm 1$. Hence $K = \ker\,\psi$. We obtain finally that

$$G/K \cong \mathfrak{S}_3$$

and thus, $|G| = |H||K| = 48$.
Moreover, as $K \lhd G$, $K \cap H = I_3$ (since I_3 is the only diagonal permutation matrix) and $G = HK$, We obtain then that

$$G \cong H \rtimes K.$$

Observe that, for all $g \in G$,

$$\det(g) = \det(P_\sigma k) = \text{sign}(\sigma)\det(k) = \pm 1\,.$$

Now consider the third subgroup $G^+ = \text{Iso}^+(X) := \text{Iso}(X) \cap SO_3(\mathbb{R}) = \{g \in G \mid \det(g) = 1\} =$, G^+ has index 2 in G and hence, is normal. Let $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ be the set of the four big diagonals of the cube X, where

$$D_1 = [AA'],\ D_2 = [BB'],\ D_3 = [CC'],\ D_4 = [DD']\,.$$

Since G fixes O, any image of two inverted points, with respect to O is again two inverted points, with respect to O. That yields to an action of G on the set $\mathcal{D}$ and thus, a morphism

$$\varphi : G \longrightarrow \mathfrak{S}(\mathcal{D}) \cong \mathfrak{S}_4\,.$$

Let $s_O := -I_3 \in G$ denotes the inversion with respect to O. Clearly, $s_O \in \ker\,\varphi$.
Let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad R_{\frac{\pi}{2}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Geometrically, T is the orthogonal symmetry with respect to the plane of equation $y = z$, and $R_{\frac{\pi}{2}}$ the rotation of 90° and of axis Oz. We clearly see that T fixes C, C', D, D' and permutes $A \leftrightarrow B'$ and $B \leftrightarrow A'$, while R permutes circularly the vertices A, B, C, D. $\varphi(T)$ is hence the transposition $(D_1 D_2)$ and $\varphi(R_{\frac{\pi}{2}})$ the *4-cycle* $(D_1 D_2 D_3 D_4)$.

Since transpositions and *4-cycles* generate $\mathfrak{S}(\mathfrak{D})$, we deduce that φ is surjective and thus

$$G/\ker\varphi \cong \mathfrak{S}_4$$

and $|\ker\varphi| = \frac{|G|}{|\mathfrak{S}_4|} = 2$. Hence

$$\ker\varphi = \{\text{id}, s_O\} = \langle s_O \rangle \lhd G$$

As $s_O \in G \backslash G^+$ (since $\det(s_O) = -1$), the restriction $\varphi_{|G^+}$ is injective, and thus, by a cardinality argument

$$G^+ \cong \mathfrak{S}_4\,.$$

We finally obtain

$$\text{Iso}(X) = \text{Iso}(X)^+ \times \langle s_O \rangle \cong \mathfrak{S}_4 \times \mathbb{Z}_2\,.$$

That brings the study of $\mathrm{Iso}(X)$ into the study of its much smaller subgroup $\mathrm{Iso}(X)^+$. Since $\mathrm{Iso}(X)^+$ is isomorphic to $\mathfrak{S}_4$, $\mathrm{Iso}(X)^+$ has 5 conjugacy classes, which we list below :

- $C_1 = \{\mathrm{id}\}$
- $C_2 = \{(D_1D_2), (D_1D_3), (D_1D_4), (D_2D_3), (D_2D_4), (D_3D_4)\}$
- $C_3 = \{(D_1D_2D_3), (D_1D_3D_2), (D_1D_2D_4), (D_1D_4D_2), (D_1D_3D_4), (D_1D_4D_3), (D_2D_3D_4), (D_2D_4D_3)\}$
- $C_4 = \{(D_1D_2D_3D_4), (D_1D_2D_4D_3), (D_1D_3D_2D_4), (D_1D_3D_4D_2), (D_1D_4D_2D_3), (D_1D_4D_3D_2)\}$
- $C_5 = \{(D_1D_2)(D_3D_4), (D_1D_3)(D_2D_4), (D_1D_4)(D_2D_3)\}$

Their correspondent matrices :

$$
\begin{aligned}
C_1 &= \left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right\}\\
C_2 &= \left\{\begin{pmatrix}-1&0&0\\0&0&-1\\0&-1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&-1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\}\\
C_3 &= \left\{\begin{pmatrix}0&0&-1\\-1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\right.\\
&\qquad \left.\begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&-1\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix}\right\}\\
C_4 &= \left\{\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\-1&0&0\\0&0&1\end{pmatrix}\right\}\\
C_5 &= \left\{\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\}
\end{aligned}
$$

And their geometric interpretation :

- The identity.
- 6 orthogonal symmetries with respect to the planes of equations : $x = \pm y, x = \pm z, y = \pm z$ (as seen above).
- 8 rotations around the four diagonal axis D_i of angle $\pm\frac{2\pi}{3}$.
- 6 rotations, respectively, around the axes Ox, Oy, Oz of angle $\pm\frac{\pi}{2}$ (as seen above)
- 3 rotations respectively, around the axes Ox, Oy, Oz of angle π (Which also are, the square of, respectively, the 6 previous rotations).

We shall finally construct the character table of $\mathrm{Iso}(X)$. Since $\mathrm{Iso}(X) = \mathrm{Iso}(X)^+ \times \{\mathrm{id}, s_O\}$, the conjugacy classes of $\mathrm{Iso}(X)$ are $\{\pm C_1, \pm C_2, \pm C_3, \pm C_4, \pm C_5\}$. We can already list the irreducible representations det, sign along with the standard representation of $\mathrm{Iso}(X)$ in $\mathbb{C}^3$, and their products.

Iso(X)	C_1	C_2	C_3	C_4	C_5	$-C_1$	$-C_2$	$-C_3$	$-C_4$	$-C_5$
1	1	1	1	1	1	1	1	1	1	1
det	1	1	1	1	1	-1	-1	-1	-1	-1
sign	1	-1	1	-1	1	1	-1	1	-1	1
det ·sign	1	-1	1	-1	1	-1	1	-1	1	-1
	n									
	m									
χ_{std}	3	-1	0	1	-1	-3	1	0	-1	1
det $\cdot\chi_{\text{std}}$	3	-1	0	1	-1	3	-1	0	1	-1
sign $\cdot\, \chi_{\text{std}}$	3	1	0	-1	-1	-3	-1	0	1	1
det ·sign $\cdot\, \chi_{\text{std}}$	3	1	0	-1	-1	3	1	0	-1	-1

From Burnside's Formula, we have that

$$1^2 + 1^2 + 1^2 + 1^2 + n^2 + m^2 + 3^2 + 3^2 + 3^2 + 3^2 = n^2 + m^2 + 40 = 48\,.$$

The only possible values for n and m are $n = m = 2$. As $\text{Iso}(X)/K \cong \mathfrak{S}_3$, the 2-dimensional irreducible representation of $\mathfrak{S}_3$ is extended by ψ to an irreducible representation of Iso(X), with denoted character χ'. We notice by looking at the product sign $\cdot\,\chi'$, that we get back to χ', while the product det $\cdot\chi'$ provides the missing character.

Iso(X)	C_1	C_2	C_3	C_4	C_5	$-C_1$	$-C_2$	$-C_3$	$-C_4$	$-C_5$
1	1	1	1	1	1	1	1	1	1	1
det	1	1	1	1	1	-1	-1	-1	-1	-1
sign	1	-1	1	-1	1	1	-1	1	-1	1
det ·sign	1	-1	1	-1	1	-1	1	-1	1	-1
χ'	2	0	-1	0	2	2	0	-1	0	2
det $\cdot\chi'$	2	0	-1	0	-2	-2	0	1	0	2
χ_{std}	3	-1	0	1	-1	-3	1	0	-1	1
det $\cdot\chi_{\text{std}}$	3	-1	0	1	-1	3	-1	0	1	-1
sign $\cdot\, \chi_{\text{std}}$	3	1	0	-1	-1	-3	-1	0	1	1
det ·sign $\cdot\, \chi_{\text{std}}$	3	1	0	-1	-1	3	1	0	-1	-1

The standard representation of $\mathcal{S}_4$, by permutation of the basis elements of $\mathbb{C}^4$, is the direct sum of the trivial representation and an irreducible representation $\tilde{\pi}$. By extending it to Iso(X) by φ, we obtain an irreducible representation, with character det ·sign $\cdot\, \chi_{\text{std}}$.